Methods and Formulas for the Stata Module **svarih**

Daniel C. Schneider

September 5, 2014

1 General Remarks

The formulas shown in this document have been chosen for easiest analytical exposition. For computational reasons, the actual calculations performed by **svarih** are frequently based on rewritten but equivalent expressions.

Regarding notation, all covariance matrices are denoted by Σ with a subscript that determines the parameters Σ refers to. In general, the notation in this document follows largely the one in **[TS] var svar** and **[TS] irf create**, except for the following differences:

u_t VAR residuals
n # of model equations

There are other minor differences but they should be self-explanatory. For example, G in **[TS] irf create** is denoted by $\breve{G}$ here.

Additional notation is as follows:

s regime index
$\bar{s}$ # of regimes in the model
T_s # of obs in regime s
K commutation matrix

s depends on time t which can be written explicitly as $s(t)$ but for notational simplicity this is suppressed in this document.

2 IH Model Setup

2.1 IH-BAC

The basic equation of the generalized Bacchiocchi model is

$$Au_t = \underbrace{(B + ED_s)}_{C_s} e_t \ , \quad e_t \sim N(0, I_n) \tag{1}$$

The time-varying $n \times n$ matrix D_s determines the volatility regime. It is a diagonal matrix whose diagonal entries consists of zeroes or ones which means that it switches columns of E on or off. Following Bacchiocchi (2011b), we make some definitions and transformations in order to write the model in a compact form that encompasses all states. This will be useful for later sections. Let Y be the $T \times n$ matrix of data on the endogenous variables. Define

$$u^* = \left(i'_{\bar{s}} \otimes u\right) \odot \left(\bar{P} \otimes i'_n\right)$$

with $\bar{P}$ being a $T \times \bar{s}$ matrix whose t, s-th element is equal to one when the system is in state s at time t and zero otherwise. $i_{\bar{s}}$ and i_n are vectors of ones of length $\bar{s}$ and n, respectively, and $\odot$ is the Hadamard product. To give an example, if $T = 3$, $\bar{s} = 2$, and $n = 2$, and $s(t = 1) = s(t = 3) = 1$ and $s(t = 2) = 2$ we would have

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which for a bivariate system translates into a residual matrix of

$$u^* = \begin{bmatrix} u_{1,1} & u_{1,2} & 0 & 0 \\ 0 & 0 & u_{2,1} & u_{2,2} \\ u_{3,1} & u_{3,2} & 0 & 0 \end{bmatrix}$$

where the first subscript denotes time and the second indexes the equation. Define an error matrix e^* of the same dimension in an analogous manner. u_t^* and e_t^* denote the t-th rows of these matrices. Furthermore, let coefficient matrices with an asterisk denote block-diagonal matrices whose $\bar{s}$ blocks are composed of the coefficient matrices A, B, and E

$$\begin{aligned} A^* &= (I_{\bar{s}} \otimes A) \\ B^* &= (I_{\bar{s}} \otimes B) \\ E^* &= (I_{\bar{s}} \otimes E) \end{aligned}$$

and let

$$D = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_{\bar{s}} \end{bmatrix}$$

denote a diagonal matrix whose s-th diagonal block contains the (diagonal) column selection matrix D_s of state s. After these definitions, we can write the model in compact form as

$$A^* u_t^* = \underbrace{(B^* + E^* D)}_{C^*} e_t^* \tag{2}$$

The model parameters are

$$\theta = \left(\text{vec}\,[A]' \quad \text{vec}\,[B]' \quad \text{vec}\,[E]'\right) \tag{3}$$

and the concentrated log-likelihood is

$$\begin{aligned} l(\theta) &= -\frac{Tn}{2} \log 2\pi + T \log |A| - \sum_T \log |B + ED_s| \\ & -\frac{1}{2} \sum_T \text{tr} \left[u_t^* u_t^{*\prime} A^{*\prime} (B^* + E^* D)^{-1\prime} (B^* + E^* D)^{-1} A^* \right] \end{aligned} \tag{4}$$

2.2 IH-BFA

Per Bacchiocchi and Fanelli (2012), the model equations for a model of up to 4 regimes are

$$\begin{aligned} u_t &= Be_t & s = 1 \\ u_t &= (B + E_s)\, e_t & 2 \leq s \leq 4 \end{aligned} \tag{5}$$

with $e_t \sim N(0, I_n)$. The vector of model parameters is

$$\theta = \left(\text{vec}\,[B]' \quad \text{vec}\,[E_2]' \quad \dots \quad \text{vec}\,[E_{\bar{s}}]'\right) \tag{6}$$

and the concentrated log-likelihood is

$$\begin{aligned} l(\theta) &= const - T_1 \log |B| - \sum_{s=2}^{\bar{s}} T_s \log |B + E_s| \\ & -\frac{T_1}{2} \text{tr} \left[\hat{\Sigma}_{u,1} B^{-1\prime} B^{-1} \right] - \sum_{s=2}^{\bar{s}} \frac{T_s}{2} \text{tr} \left[\hat{\Sigma}_{u,s} (B + E_s)^{-1\prime} (B + E_s)^{-1} \right] \end{aligned} \tag{7}$$

2.3 IH-LLU

Per Lanne and Lütkepohl (2008), the model equations for a model with two regimes are

$$\begin{aligned} u_t &= Be_t & s=1 \\ u_t &= BL^{0.5}e_t & s=2 \end{aligned} \tag{8}$$

with $e_t \sim N(0, I_n)$. The vector of model parameters is

$$\theta = \left(\operatorname{vec}[B]' \quad \operatorname{vec}[\operatorname{diag}[L]]'\right) \tag{9}$$

where $\operatorname{diag}[\ldots]$ extracts the diagonal elements of a square matrix, and the concentrated log-likelihood is

$$\begin{aligned} l(\theta) &= -\frac{Tn}{2}\log 2\pi - \frac{T_1}{2}\left(\log|BB'| + \operatorname{tr}\left[\hat{\Sigma}_{u,1}B^{-1'}B^{-1}\right]\right) \\ &\quad -\frac{T_2}{2}\left(\log|BLB'| + \operatorname{tr}\left[\hat{\Sigma}_{u,2}B^{-1'}L^{-1}B^{-1}\right]\right) \end{aligned} \tag{10}$$

3 Identification

3.1 IH-BAC

The following proposition follows a similar one from Bacchiocchi (2011a), extended for the additional E-matrix in the model implemented in **svarih bacchiocchi**.[1]

Proposition 1 *Let the DGP follow the process (1) featuring $\bar{s} \geq 2$ states of volatility, and suppose that the DGP admits linear restrictions of the form $R_A \operatorname{vec}[A] = r_A$ and similarly for the coefficient matrices B and E. Then, (A_0, B_0, E_0) are locally identified if and only if the matrix*

$$\begin{bmatrix} -2N_{(n\bar{s})}\bar{Q}_1\bar{H} & 2N_{(n\bar{s})}\bar{Q}_2\bar{H} & 2N_{(n\bar{s})}\bar{Q}_3\bar{H} \\ R_A & 0 & 0 \\ 0 & R_B & 0 \\ 0 & 0 & R_E \end{bmatrix}$$

has full column rank $3n^2$, where the matrices $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ are defined by

$$\begin{aligned} \bar{Q}_1 &= \left(A^{*-1}(B^* + E^*D)(B^* + E^*D)' A^{*-1'} \otimes A^{*-1}\right) \\ \bar{Q}_2 &= \left(A^{*-1}(B^* + E^*D) \otimes A^{*-1}\right) \\ \bar{Q}_3 &= \left(A^{*-1}(B^* + E^*D)D \otimes A^{*-1}\right) \end{aligned}$$

and

$$\begin{aligned} N_{(n\bar{s})} &= \frac{1}{2}\left(I_{n^2\bar{s}^2} + K_{(n\bar{s})}\right) \\ \bar{H} &= \left(I_{\bar{s}} \otimes K_{n,\bar{s}}\right)\left(\operatorname{vec}[I_{\bar{s}}] \otimes I_n\right) \otimes I_n \end{aligned}$$

3.2 IH-BFA

Write the constraints of the model in explicit form as

$$\theta = S \cdot \gamma + s$$

[1]Derivations are available upon request.

See **[P] makecns** for more information on converting constraints from implicit to explicit form and vice versa. Per Bacchiocchi and Fanelli (2012), the estimates are locally identified if and only if the matrix

$$\left(I_{\bar{s}} \otimes D_n^+\right) \begin{bmatrix} (C \otimes I_n) & & & \\ (C+E_2) \otimes I_n & (C+E_2) \otimes I_n & & \\ \vdots & \vdots & \ddots & \\ (C+E_{\bar{s}}) \otimes I_n & & & (C+E_{\bar{s}}) \otimes I_n \end{bmatrix} \cdot S$$

has full column rank, where $D_n^+ = \left(D_n' D_n\right)^{-1} D_n'$ and D_n is the duplication matrix.

3.3 IH-LLU

A check for local identification is not implemented in **svarih llutkepohl**.

Per Lanne and Lütkepohl (2008), coefficients are globally identified (up to sign reversals of columns of model matrices) if the diagonal elements of diagonal matrix L are distinct. **svarih** performs pairwise Wald tests for equality of the elements of L, and returns the minimum of these Wald statistics, along with its associated p-value.

4 Gradients and Hessians

This section states the first and second derivatives of the concentrated log-likelihood functions which are used in the ML optimization algorithm.

4.1 IH-BAC

Write U_t^* and U^* for $u_t^* u_t^{*'}$ and $\sum_T u_t^* u_t^{*'}$, respectively. The following results are similar to proposition 2 of Bacchiocchi (2011a), extended for the additional E-matrix in the model implemented in **svarih bacchiocchi**.[2]

The gradient of the likelihood (4) expressed as a row vector is given by

$$\begin{aligned} g(\theta) &= \left[g_A(\theta) \quad g_B(\theta) \quad g_E(\theta)\right] \\ g_A(\theta) &= -\operatorname{vec}\left[C^{*-1'} C^{*-1} A^* U^*\right]' \bar{H} + T \operatorname{vec}\left[A^{-1'}\right]' \\ g_B(\theta) &= \operatorname{vec}\left[C^{*-1'} C^{*-1} A^* U^* A^{*'} C^{*-1'} - C^{*-1'} T^*\right]' \bar{H} \\ g_E(\theta) &= \operatorname{vec}\left[C^{*-1'} C^{*-1} A^* U^* A^{*'} C^{*-1'} D - C^{*-1'} D T^*\right]' \bar{H} \end{aligned}$$

Recall that $\bar{H} = (I_{\bar{s}} \otimes K_{n,\bar{s}}) (vec(I_{\bar{s}}) \otimes I_n) \otimes I_n$. We define $\bar{H}_3 = I_3 \otimes \bar{H}$. The Hessian is

$$\tilde{F} = \bar{H}_3' \tilde{F}_1 \bar{H}_3 + \tilde{F}_2, \quad \tilde{F}_1 = \begin{bmatrix} \tilde{F}_1^{AA} & \tilde{F}_1^{AB} & \tilde{F}_1^{AE} \\ \tilde{F}_1^{BA} & \tilde{F}_1^{BB} & \tilde{F}_1^{BE} \\ \tilde{F}_1^{EA} & \tilde{F}_1^{EB} & \tilde{F}_1^{EE} \end{bmatrix}, \quad \tilde{F}_2 = \begin{bmatrix} \tilde{F}_2^{AA} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[2]Derivations are available upon request.

where the individual terms are

$$
\begin{aligned}
\tilde{F}_1^{AA} &= -\left(\tilde{U}^* \otimes C^{*-1'} C^{*-1}\right) \\
\tilde{F}_1^{AB} &= \left(\tilde{U}^* A^{*'} C^{*-1'} C^{*-1} \otimes C^{*-1'}\right) K_{(n\bar{s})} + \left(\tilde{U}^* A^{*'} C^{*-1'} \otimes C^{*-1'} C^{*-1}\right) \\
\tilde{F}_1^{AE} &= \left(\tilde{U}^* A^{*'} C^{*-1'} C^{*-1} \otimes C^{*-1'} D\right) K_{(n\bar{s})} + \left(\tilde{U}^* A^{*'} C^{*-1'} D \otimes C^{*-1'} C^{*-1}\right) \\
\tilde{F}_1^{BB} &= \left(T^* C^{*-1} \otimes C^{*-1'}\right) K_{(n\bar{s})} - \left(C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'} \otimes C^{*-1'} C^{*-1}\right) \\
&\quad -2\left(C^{*-1} \otimes C^{*-1'} C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'}\right) K_{(n\bar{s})} \\
\tilde{F}_1^{BE} &= \left(T^* C^{*-1} \otimes C^{*-1'} D\right) K_{(n\bar{s})} - \left(C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'} C^{*-1} \otimes C^{*-1'} D\right) K_{(n\bar{s})} \\
&\quad -\left(C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'} D \otimes C^{*-1'} C^{*-1}\right) - \left(C^{*-1} \otimes C^{*-1'} C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'} D\right) K_{(n\bar{s})} \\
\tilde{F}_1^{EE} &= \left(T^* D C^{*-1} \otimes C^{*-1'} D\right) K_{(n\bar{s})} - \left(D C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'} D \otimes C^{*-1'} C^{*-1}\right) \\
&\quad -2\left(D C^{*-1} \otimes C^{*-1'} C^{*-1} A^* \tilde{U}^* A^{*'} C^{*-1'} D\right) K_{(n\bar{s})}
\end{aligned}
$$

and

$$\tilde{F}_2^{AA} = -T\left(A^{-1} \otimes A^{-1'}\right) K_{(n)}$$

4.2 IH-BFA

Building on Bacchiocchi and Fanelli (2012), the gradient of (7) expressed as a row vector is

$$
\begin{aligned}
g(\theta) &= \left[g_B(\theta) \quad g_{E_2}(\theta) \quad \dots \quad g_{E_{\bar{s}}}(\theta)\right] \\
g_{E_s}(\theta) &= \mathrm{vec}\left[-T_s (B+E)^{-1'} + T_s (B+E)^{-1'} (B+E)^{-1} \hat{\Sigma}_{u,s} (B+E)^{-1'}\right]' \\
g_B(\theta) &= \mathrm{vec}\left[-T_1 B^{-1'} + T_1 B^{-1'} B^{-1} \hat{\Sigma}_{u,1} B^{-1'}\right]' + \sum_{s=2}^{\bar{s}} g_{E_s}(\theta)
\end{aligned}
$$

The Hessian is

$$
\tilde{F} = \begin{bmatrix}
\tilde{F}^{BB} & \tilde{F}^{BE_2} & \dots & \tilde{F}^{BE_{\bar{s}}} \\
\tilde{F}^{E_2 B} & \tilde{F}^{E_2 E_2} & & \tilde{F}^{E_2 E_{\bar{s}}} \\
\vdots & & \ddots & \vdots \\
\tilde{F}^{E_{\bar{s}} B} & \tilde{F}^{E_{\bar{s}} E_2} & \dots & \tilde{F}^{E_{\bar{s}} E_{\bar{s}}}
\end{bmatrix}
$$

where the individual terms are

$$
\begin{aligned}
\tilde{F}^{E_s E_s} &= T_s\left((B+E_s)^{-1} \otimes (B+E_s)^{-1'}\right) K_{(n)} \\
&\quad -2T_s\left((B+E_s)^{-1} \otimes (B+E_s)^{-1'} (B+E_s)^{-1} \hat{\Sigma}_{u,s} (B+E_s)^{-1'}\right) K_{(n)} \\
&\quad -T_s\left((B+E_s)^{-1} \hat{\Sigma}_{u,s} (B+E_s)^{-1'} \otimes (B+E_s)^{-1'} (B+E_s)^{-1}\right) \\
\tilde{F}^{BE_s} &= \tilde{F}^{E_s E_s} \\
\tilde{F}^{BB} &= T_1\left(B^{-1} \otimes B^{-1'}\right) K_{(n)} \\
&\quad -2T_1\left(B^{-1} \otimes B^{-1'} B^{-1} \hat{\Sigma}_{u,1} B^{-1'}\right) K_{(n)} \\
&\quad -T_1\left(B^{-1} \hat{\Sigma}_{u,1} B^{-1'} \otimes B^{-1'} B^{-1}\right) \\
&\quad +\sum_{s=2}^{\bar{s}} \tilde{F}^{E_s E_s}
\end{aligned}
$$

and $\tilde{F}^{E_i E_j}$, $i \neq j$, are null matrices.

4.3 IH-LLU

To shorten the exposition, the gradient and Hessian are shown as if the log-likelihood (10) were maximized over all elements of the diagonal matrix L.[3] Since the off-diagonal elements of L are zero, the gradient and Hessian for θ can be easily obtained by appropriately deleting rows and columns from the resulting expressions. Denoting $\tilde{\theta} = \begin{bmatrix} \text{vec}\,[B]' & \text{vec}\,[L]' \end{bmatrix}$, the gradient expressed as a row vector is

$$\begin{aligned}
g(\theta) &= \begin{bmatrix} g_B(\theta) & g_L(\theta) \end{bmatrix} \\
g_B(\theta) &= -T\,\text{vec}\left[B^{-1'}\right]' + T_1\,\text{vec}\left[B^{-1'}B^{-1}\hat{\Sigma}_{u,1}B^{-1'}\right]' + T_2\,\text{vec}\left[B^{-1'}L^{-1}B^{-1}\hat{\Sigma}_{u,2}B^{-1'}\right]' \\
g_L(\theta) &= -\frac{T_2}{2}\,\text{vec}\left[L^{-1}\right]' + \frac{T_2}{2}\,\text{vec}\left[L^{-1}B^{-1}\hat{\Sigma}_{u,2}B^{-1'}L^{-1}\right]'
\end{aligned}$$

The Hessian is

$$\tilde{F} = \begin{bmatrix} \tilde{F}^{BB} & \tilde{F}^{BL} \\ \tilde{F}^{LB} & \tilde{F}^{LL} \end{bmatrix}$$

where the individual terms are

$$\begin{aligned}
\tilde{F}^{BB} &= T\left(B^{-1} \otimes B^{-1'}\right)K_{(n)} - T_1\left(B^{-1}\hat{\Sigma}_{u,1}B^{-1'} \otimes B^{-1'}B^{-1}\right) - 2T_1\left(B^{-1} \otimes B^{-1'}B^{-1}\hat{\Sigma}_{u,1}B^{-1'}\right)K_{(n)} \\
&\quad -T_2\left(B^{-1}\hat{\Sigma}_{u,2}B^{-1'} \otimes B^{-1'}L^{-1}B^{-1}\right) - 2T_2\left(B^{-1} \otimes B^{-1'}L^{-1}B^{-1}\hat{\Sigma}_{u,2}B^{-1'}\right)K_{(n)} \\
\tilde{F}^{BL} &= -T_2\left(B^{-1}\hat{\Sigma}_{u,2}B^{-1'}L^{-1} \otimes B^{-1'}L^{-1}\right) \\
\tilde{F}^{LL} &= \frac{T_2}{2}\left(L^{-1} \otimes L^{-1}\right) - T_2\left(L^{-1} \otimes L^{-1}B^{-1}\hat{\Sigma}_{u,2}B^{-1'}L^{-1}\right)
\end{aligned}$$

5 GLS-VAR Calculations

For IH-BAC and IH-LLU, option **glsiter()** invokes an iteration between ML estimation of contemporaneous model parameters and GLS-VAR estimation of the VAR slope coefficients. Following Lanne and Lütkepohl (2008), let Z_t denote the (column) vector of all VAR regressor variables at time t, and let $\tilde{M}$ denote the matrix of all VAR slope coefficients. The GLS-VAR coefficients are calculated as

$$\text{vec}\left[\tilde{M}\right] = \left[\sum_{s=1}^{\bar{s}}\sum_{t\in s}\left(Z_t Z_t' \otimes \hat{\Sigma}_{u,s}^{-1}\right)\right]^{-1}\left[\sum_{s=1}^{\bar{s}}\sum_{t\in s}\left(Z_t \otimes \hat{\Sigma}_{u,s}^{-1}\right)\right]$$

with the first expression in brackets being their covariance matrix estimate. $\hat{\Sigma}_{u,s}$ is based on the ML estimates of the current iteration. In particular

$$\hat{\Sigma}_{u,s} = \hat{A}^{-1}\left(\hat{B} + \hat{E}D_s\right)\left(\hat{B} + \hat{E}D_s\right)'\hat{A}^{-1'}$$

for IH-BAC and

$$\begin{aligned}
\hat{\Sigma}_{u,1} &= \hat{B}\hat{B}' \\
\hat{\Sigma}_{u,2} &= \hat{B}\hat{L}\hat{B}'
\end{aligned}$$

for IH-LLU.

6 Conditional Structural IRFs and Structural FEVDs

6.1 SVAR Formulas

The following reproduces the standard formulas for SIRFs and SFEVDs from **[TS] irf create**, with some minor deviations in notation and presentation.

[3]Derivations are available upon request.

6.1.1 Impulse Response Functions

Let $\hat{A}_i$, $i = 1, \ldots, p$ denote the estimated VAR lag coefficient matrices. The estimated coefficient matrices of the vector moving average representation are calculated as

$$\hat{\Phi}_h = \sum_{j=1}^{h} \hat{\Phi}_{h-j}\hat{A}_h$$

Since we consider a standard AB-model, we define $\hat{P} = \hat{A}^{-1}\hat{B}$ and obtain structural IRFs by

$$\hat{\Theta}_h = \hat{\Phi}_h \hat{P}$$

In order to write down formulas for the covariance matrix of the IRFs, we have to make several additional definitions. Let $\hat{M}$ denote the estimated VAR companion matrix and

$$\begin{aligned}
J &= \begin{pmatrix} I_n & 0_n & \cdots & 0_n \end{pmatrix} \\
\hat{\Sigma}_{AB} &= \widehat{\text{cov}}\left[\text{vec}\left[\hat{A}\right]' \quad \text{vec}\left[\hat{B}\right]'\right]' \\
\hat{\Sigma}_{\Pi} &= \widehat{\text{cov}}\left[\text{vec}\left[\hat{\Pi}\right]\right] \quad \text{with } \hat{\Pi} = \begin{pmatrix} \hat{A}_1 & \cdots & \hat{A}_p \end{pmatrix} \\
\breve{G}_0 &= 0_n \\
\breve{G}_i &= \sum_{k=0}^{i-1} \left\{ \hat{P}'J\left(\hat{M}'\right)^{i-1-k} \otimes \left(J\hat{M}^k J'\right) \right\} \\
\bar{Q} &= \left(\hat{P}' \otimes \hat{P}\right) \left\{ \; I_n \otimes \hat{B}^{-1} \qquad -\hat{P}^{-1'} \otimes \hat{B}^{-1} \; \right\} \\
\hat{\Sigma}(0) &= \bar{Q}\hat{\Sigma}_{AB}\bar{Q}'
\end{aligned}$$

Then the covariance matrix for the impulse response matrix at response step h is obtained as the h-th, h-th $n^2 \times n^2$ block of the block-wise defined matrix

$$\hat{\Sigma}\left(\bar{h}\right)_{ij} = \breve{G}_i \hat{\Sigma}_{\Pi} \breve{G}_j + \left(I_n \otimes J\hat{M}^i J'\right) \hat{\Sigma}(0) \left(I_n \otimes J\hat{M}^j J'\right)'$$

with $\bar{h}$ being the maximum response step.

6.1.2 Forecast Error Variance Decompositions

Using the definitions

$$\begin{aligned}
\bar{F}_h &= \left(\sum_{i=0}^{h-1} \hat{\Theta}_i \hat{\Theta}_i'\right) \odot I_n \\
\bar{M}_h &= \sum_{i=0}^{h-1} \hat{\Theta}_i \odot \hat{\Theta}_i
\end{aligned}$$

the FEVD matrix at response step h is

$$\bar{W}_h = \bar{F}_h^{-1} \bar{M}_h$$

Letting $\bar{D}_{\breve{Q}}$ denote a diagonal matrix with diagonal elements equal to $\text{vec}\left[\breve{Q}\right]$, $\breve{Q}$ being an arbitrary matrix, and using the previous definitions of $\hat{\Sigma}\left(\bar{h}\right)$ and N_n, and

$$\begin{aligned}
\frac{\partial \, \text{vec}\left[\bar{W}_h\right]}{\partial \, \text{vec}\left[\hat{\Theta}_j\right]} &= 2\left\{ \left(I_n \otimes \bar{F}_h^{-1}\right) \bar{D}_{\hat{\Theta}_j} - \left(\bar{W}_h' \otimes \bar{F}_h^{-1}\right) \bar{D}_{I_n} N_n \left(\hat{\Theta}_j \otimes I_n\right) \right\} \\
\bar{Z}_h &= \begin{pmatrix} \frac{\partial \, \text{vec}\left[\bar{W}_h\right]}{\partial \, \text{vec}\left[\hat{\Theta}_0\right]} & \cdots & \frac{\partial \, \text{vec}\left[\bar{W}_h\right]}{\partial \, \text{vec}\left[\hat{\Theta}_{\bar{h}}\right]} \end{pmatrix}
\end{aligned}$$

we obtain the asymptotic covariance matrix of $\text{vec}\left[\bar{W}_h\right]$ as

$$\bar{Z}_h \hat{\Sigma}\left(\bar{h}\right) \bar{Z}_h'$$

It is implicitly understood that $\frac{\partial\, \text{vec}\left[\bar{W}_h\right]}{\partial\, \text{vec}\left[\hat{\Theta}_j\right]}$ resolves to a null matrix if $j \geq k$.

6.2 Adjustment for the IH Setting

Conditional SIRFs/SFEVDs for IH methods use identical formulas as SVARs but certain elements of the SVAR formulas have to be calculated differently. These are $\hat{M}$, $\hat{\Phi}_h$, $\hat{\Sigma}_{\hat{\Pi}}$, $\hat{P}$, and $\hat{\Sigma}_{AB}$. The first three of these magnitudes concern the VAR slope coefficients and their covariance matrices. For IH-BAC and IH-LLU, if option **glsiter()** is not used or if option **glsiter(0)** is specified, these are based on the underlying VAR. If **glsiter(#)**, #>0, is used, they are based on the underlying GLS-VAR. For IH-BFA, the three magnitudes receive a subscript s and are based on the $\bar{s}$ underlying subsample VARs.

The adjustment of $\hat{P} = \hat{A}^{-1}\hat{B}$ and $\hat{\Sigma}_{AB}$ in the standard SVAR formulas concerns the contemporaneous ML parameters and their covariances. In SVAR models, the matrix A models the contemporaneous interactions between endogenous variables and the matrix B models the contemporaneous impact of shocks on the endogenous variables. In the IH models presented here, the shock impact matrix varies over regimes and is composed of the elements of B and E (IH-BAC), B and E_s, $2 \leq s \leq 4$ (IH-BFA), and B and L (IH-LLU), so the notational symbol B takes on a different meaning. It stands for the shock impact matrix in what can be called the baseline volatility state. Let's denote the regime-dependent shock impact matrix for IH models C_s. What we are seeking in order to utilize standard SVAR formulas for the calculation of conditional SIRFs/SFEVDs of IH models and their standard errors is $\hat{P}_s = \hat{A}^{-1}\hat{C}_s$ and $\hat{\Sigma}_{AC_s}$.

IH-BAC In state s, the relevant coefficients are those of A and C_s, the latter being a linear combination of B and E:

$$C_s = B + ED_s$$

and hence $\hat{P}_s = \hat{A}^{-1}\hat{C}_s$. In vec notation, the equation for C_s reads

$$\begin{aligned} \text{vec}\left[C_s\right] &= \text{vec}\left[B\right] + \text{vec}\left[ED_s\right] \\ &= I_{n^2}\, \text{vec}\left[B\right] + \left(D_s \otimes I_n\right) \text{vec}\left[E\right] \end{aligned}$$

Since the estimates of (3) are normally distributed, any linear combination of the parameters is again normally distributed. Writing

$$\theta^{AC} = \begin{bmatrix} \text{vec}\left[A\right]' & \text{vec}\left[C_s\right]' \end{bmatrix}$$

the application of standard normal distribution theory immediately implies

$$\hat{\theta}^{AC_s} \xrightarrow{d} N\left(\theta_0^{AC_s}, \Sigma_{AC_s}\right)$$

with

$$\Sigma_{AC_s} = \bar{G}_s \Sigma_\theta \bar{G}_s', \quad \bar{G}_s = \begin{bmatrix} I_{n^2} & 0 & 0 \\ 0 & I_{n^2} & D_s \otimes I_n \end{bmatrix}$$

IH-BFA Since $A = I_n$, the first block row and the first block column of Σ_{AC_s} consist of null matrices. In state 1 we have $C_1 = B$. Since the estimates of (6) follow a normal distribution, we can simply cut out the block regarding $\text{vec}\left[\hat{B}\right]'$ from $\hat{\Sigma}_\theta$ and plug it into $\hat{\Sigma}_{AC_1}$. In states $2 \leq s \leq 4$, the required linear combination of $C_s = B + E_s$ leads to the normally distributed vector $\tilde{\theta}' = \left[I_{n^2} \;\; \bar{e}_s' \otimes I_{n^2}\right] \hat{\theta}'$, where $\bar{e}_s'$ stands for a unit row vector with element s equal to one. The associated covariance matrix is

$$\Sigma_{\tilde{\theta}} = \bar{G} \Sigma_\theta \bar{G}' \;, \quad \bar{G} = \left[I_{n^2} \;\; \bar{e}_s' \otimes I_{n^2}\right]$$

IH-LLU As in IH-BFA, we have $A = I_n$ and $C_1 = B$ in state 1, and the same comments as for IH-BFA apply. For state 2, we need to calculate

$$B \cdot L^{0.5} = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \cdot \begin{bmatrix} l_1^{0.5} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_n^{0.5} \end{bmatrix}$$

The appropriate covariance matrix is obtained via the delta method as

$$\Sigma_{\tilde{\theta}} = \bar{G}\Sigma_\theta \bar{G}'$$

using

$$\begin{aligned} \bar{G} &= \frac{\partial \operatorname{vec}\left[B \cdot L^{0.5}\right]}{\partial \theta} = \begin{bmatrix} \frac{\partial \operatorname{vec}\left[B \cdot L^{0.5}\right]}{\partial \operatorname{vec}[B]'} & \frac{\partial \operatorname{vec}\left[B \cdot L^{0.5}\right]}{\partial \operatorname{vec}[\operatorname{diag}[L]]'} \end{bmatrix} \\ \frac{\partial \operatorname{vec}\left[B \cdot L^{0.5}\right]}{\partial \operatorname{vec}[B]'} &= L^{0.5} \otimes I_n \\ \frac{\partial \operatorname{vec}\left[B \cdot L^{0.5}\right]}{\partial \operatorname{vec}[\operatorname{diag}[L]]'} &= 0.5 \cdot \begin{bmatrix} b_{11}l_1^{-0.5} & 0 & 0 \\ \vdots & & \\ b_{n1}l_1^{-0.5} & & \vdots \\ 0 & \ddots & 0 \\ \vdots & & b_{1n}l_n^{-0.5} \\ & & \vdots \\ 0 & 0 & b_{nn}l_n^{-0.5} \end{bmatrix} \end{aligned}$$

7 Calculation of Shocks

predict calculates shock series from the residuals as follows:

From (1), IH-BAC uses

$$\hat{e}_t = \left(\hat{B} + \hat{E}D_s\right)^{-1} \hat{A}\hat{u}_t$$

From (5), IH-BFA uses

$$\begin{aligned} \hat{e}_t &= \hat{B}^{-1}\hat{u}_t & s = 1 \\ \hat{e}_t &= \left(\hat{B} + \hat{E}_s\right)^{-1} \hat{u}_t & 2 \le s \le 4 \end{aligned}$$

From (8), IH-LLU uses

$$\begin{aligned} \hat{e}_t &= \hat{B}^{-1}\hat{u}_t & s = 1 \\ \hat{e}_t &= \left(\hat{B}\hat{L}^{0.5}\right)^{-1} \hat{u}_t & s = 2 \end{aligned}$$

8 Historical Decompositions

predict with its **hdecomp** option calculates a simple dynamic forecast. If the **svarih** estimation is based on ML-GLS iterations, the GLS-VAR coefficients are used, and otherwise VAR coefficients. If in addition option **hdshock()** is used and the argument refers to the shock of equation j, the dynamic prediction is augmented in each period t by the contribution of shock j to the residual of equation i. The contribution depends on the regime that the observations at time t belong to. For IH-BAC, the contribution of shock j to the residual of equation i is the i-th element of

$$\hat{A}^{-1}\left(\hat{B}+\hat{E}D_s\right)\hat{e}_{j,t}$$

where $e_{j,t}$ is the vector of implied shock series with all elements except the j-th one set to zero. For IH-BFA, analogous reasoning is applied to the equations

$$\begin{aligned} \hat{B}\hat{e}_{j,t} \qquad & s = 1 \\ \left(\hat{B}+\hat{E}_s\right)\hat{e}_{j,t} \qquad & 2 \leq s \leq 4 \end{aligned}$$

and similarly for IH-LLU

$$\begin{aligned} \hat{B}\hat{e}_{j,t} \qquad & s = 1 \\ \hat{B}\hat{L}^{0.5}\hat{e}_{j,t} \qquad & s = 2 \end{aligned}$$

References

Bacchiocchi, E. (2011a): Identification in Structural VAR Models with Different Volatility Regimes, Working Paper No. 2011-39, Universita Degli Studi di Milano.

Bacchiocchi, E. (2011b): Identification through Heteroskedasticity: A Likelihood-based Approach, Working Paper No. 2010-38, Universita Degli Studi di Milano.

Bacchiocchi, E. and L. Fanelli (2012): Identification in Structural Vector Autoregressive Models with Structural Changes, Working Paper No. 2012-16, Universita Degli Studi di Milano.

Lanne, M. and H. Lütkepohl (2008): Identifying Monetary Policy Shocks via Changes in Volatility, *Journal of Money, Credit and Banking*, 40 (6), 1131–1149.